International Journal of Theoretical Physics, Vol. 41, No. 10, October 2002 (© 2002)

Unitary Operator of $su_q(n)$-Covariant Oscillator Algebra

W.-S. Chung[1]

Received May 16, 2002

The unitary operator of $su_q(n)$-covariant oscillator algebra is constructed and two types of q-coherent states are obtained explicitly.

KEY WORDS: unitary operator; oscillator algebra.

Since the q-deformation of a single-mode oscillator algebra (Arik and Coon, 1976; Biedenharn, 1989; Macfarlane, 1989) was known, the multimode extension has attracted much interest. The development of differential calculus in noncommutative (quantized) spaces enabled us to extend the single-mode q-oscillator algebra into the multimode case (Pusz and Woronowicz, 1989). The multimode q-oscillator algebra was shown to be covariant under some quantum groups such as $gl_q(n)$, $sl_q(n)$, $su_q(n)$, and so on.

In this paper, we deal with the $su_q(n)$-covariant oscillator algebra and construct its unitary operator. We use it to present two types of q-coherent states. One is the q-analogue of the Glauber-type coherent states and the other of the Perelomov-type.

The $su_q(n)$-covariant oscillator algebra is defined by

$$a_i a_j = q a_j a_i \quad (i < j),$$

$$a_i^\dagger a_j^\dagger = q^{-1} a_j^\dagger a_i^\dagger \quad (i < j),$$

$$a_i a_j^\dagger = q a_j^\dagger a_i \quad (i \neq j),$$

$$a_i a_i^\dagger = 1 + q^2 a_i^\dagger a_i + (q^2 - 1) \sum_{k=1}^{i-1} a_k^\dagger ak,$$

[1] Department of Physics, Research Institute of Natural Science, Gyeongsang National University, Jinju 660–701, South Korea.

$$[N_i, a_j^\dagger] = \delta_{ij} a_j^\dagger,$$
$$[N_i, a_j] = -\delta_{ij} a_j, \tag{1}$$

where the deformation parameter q is assumed to be real. Algebra (1) is invariant under the hermitian conjugation, and so $a_i^\dagger$ can be interpreted as the conjugation operator of a_i and N_i is hermitian. The proof of $su_q(n)$-covariance of this algebra and its Fock representation is given in Jagannathan *et al.* (1992). Using the Fock representation given in Jagannathan *et al.* (1992), the relation between number operators and step operators is given by

$$a_i^\dagger a_i = q^{2\sum_{k=1}^{i-1} N_k} [N_i], \tag{2}$$

where the q-number is defined by

$$[x] = \frac{q^{2x} - 1}{q^2 - 1}.$$

Using relation (2), the forth relation can be rewritten as

$$[a_i, a_i^\dagger] = \prod_{k=1}^{i-1} Q_k, \tag{3}$$

where the scale operator Q_k is defined by

$$Q_k = q^{2N_k}. \tag{4}$$

Let us introduce two types of q-deformed exponential functions as follows:

$$e_{q^2}(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]!},$$
$$E_{q^2}(x) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)} x^n}{[n]!}, \tag{5}$$

where two q-deformed exponential functions satisfy

$$E_{q^2}(x)\, e_{q^2}(-x) = 1. \tag{6}$$

By use of the definition of the q-exponential function and relation (3), we have the following identity:

$$e_{q^2}(ta_i) a_i^\dagger = \left(a_i^\dagger + t \prod_{k=1}^{i-1} Q_k \right) e_{q^2}(ta_i). \tag{7}$$

If $LR = q^2 RL$, we have

$$e_{q^2}(R)\, e_{q^2}(L) = e_{q^2}(R + L). \tag{8}$$

Using property (8), Eq. (7) can be rewritten as

$$e_{q^2}(ta_i)\,e_{q^2}(ra_i^\dagger) = e_{q^2}(ra_i^\dagger)\,e_{q^2}\left(rt\prod_{k=1}^{i-1} Q_k\right)\,e_{q^2}(ta_i). \tag{9}$$

Now, we will find out the unitary operator for algebra (1). Some properties of q-exponential functions enable us to determine the correct form of the unitary operator:

$$U(\mathrm{w}) = U_N(w_N)U_{N-1}(w_{N-1})\cdots U_1(w_1), \tag{10}$$

where

$$U_i(w_i) = e_{q^2}^{-1/2}\left(|w_i|^2\prod_{k=1}^{i-1} Q_k\right)\,E_{q^2}(w_i a_i^\dagger)\,e_{q^2}(-\overline{w}_i a_i) \tag{11}$$

and w_i and $\overline{w}_i$ are commuting variables (ordinary complex variables). It can be easily verified that the operator $U(\mathrm{w})$ defined in Eq. (10) is unitary,

$$U(\mathrm{w})U^\dagger(\mathrm{w}) = U^\dagger(\mathrm{w})U(\mathrm{w}) = 1 \tag{12}$$

Indeed, let $|n_1,\ldots,n_N\rangle$ be the system of eigenstates of the number operators N_i obeying

$$N_i|n_1,\ldots,n_N\rangle = n_i|n_1,\ldots,n_N\rangle. \tag{13}$$

Since $a_i^\dagger$ (or a_i) plays a role of raising (or lowering) operator, relation (2) gives the matrix representation of $a_i^\dagger$ and a_i:

$$\begin{aligned} a_i^\dagger|n_1,\ldots,n_N\rangle &= q^{\sum_{k=1}^{i-1} n_k}\sqrt{[n_i+1]}|n_1,\ldots,n_{i+1},\ldots,n_N\rangle,\\ a_i|n_1,\ldots,n_N\rangle &= q^{\sum_{k=1}^{i-1} n_k}\sqrt{[n_i]}|n_1,\ldots,n_{i-1},\ldots,n_N\rangle, \end{aligned} \tag{14}$$

We determine the matrix coefficients of the unitary operator $U(\mathrm{w})$:

$$\begin{aligned} T^{n_1,\ldots,n_N}_{n'_1,\ldots,n'_N} &= \langle n_1,\ldots,n_N|U^\dagger(-w_1,\ldots,-\mathrm{w}_N)|n'_1,\ldots,n'_N\rangle\\ &= \Sigma\big\langle n_1,\ldots,n_N|U_1^\dagger(-w_1)|n_1^{(1)},\ldots,n_N^{(1)}\big\rangle\\ &\quad\times\big\langle n_1^{(1)},\ldots,n_N^{(1)}|U_2^\dagger(-w_2)|n_1^{(2)},\ldots,n_N^{(2)}\big\rangle\\ &\quad\cdots\big\langle n_1^{(N-1)},\ldots,n_N^{(N-1)}|U_N^\dagger(-w_N)|n'_1,\ldots,n'_N\big\rangle, \end{aligned} \tag{15}$$

where

$$\begin{aligned} (T^{(i)})^{n_1,\ldots,n_N}_{n'_1,\ldots,n'_N} &= \langle n_1,\ldots,n_N|U_i^\dagger(-w_i)|n'_1,\ldots,n'_N\rangle\\ &= \prod_{j\neq i}\delta_{n_j,n'_j}\,e_{q^2}^{-1/2}\left(|w_i|^2 q^{\sum_{k=1}^{i-1} n'_k}\right) \end{aligned}$$

$$\times \frac{(-)^{n'_i} \bar{w}_i^{n'_i} w_i^{ni} q^{n'_i(n'_i-1)+(n_i+n'^i)\sum_{k=1}^{i-1} n_k}}{\sqrt{[n_i]![n'_i]!}}$$

$$\times\, 2\phi_0 \left(q^{-2ni}, q^{-2n'_i}; -; \frac{q^{2\left(1+n_i-\sum_{k=1}^{i-1} nk\right)}}{(1-q^2)|w_i|^2} \right). \tag{16}$$

Here, the symbol $2\phi_0$ denotes the basic hypergeometric function (Gasper and Rhaman, 1990)

$$2\phi_0(a, b; -; x) = \sum_{k=0}^{\infty} \frac{(a; q^2)_k (b; q^2)_k (-1)^k q^{-k(k-1)}}{(q^2; q^2)_k} x^k, \tag{17}$$

where the Pochhammer q-symbol (or q-shifted factorial) is defined by

$$(a; q^2)_k = (1-a)(1-q^2 a)\cdots(1-aq^{2k-2}) \tag{18}$$

The function $2\phi_0$ in Eq. (16) is called the Charlier q-polynomial (Nikiforov *et al.*, 1985) because, for $q = 1$, it is identical to the ordinary Charlier polynomials (Granovskii and Zhedanov, 1986). In the classical limit ($q \to 1$), the transition coefficients can be expressed in terms of Charlier polynomial, and so $(T^{(i)})^{n_1,\ldots,n_N}_{n'_1,\ldots,n'_N}$ can be regarded as the q-analogue of the transition coefficient. At this stage, the physical meaning of q-transition coefficients in unclear.

From formula (14), it is possible to obtain expressions for the two types of q-coherent states. One is the q-analogue of Glauber-type coherent state and another q-analogue of Perelomov-type coherent state. The q-analogue of Glauber-type coherent state is defined by

$$a_i |w_1, \ldots, w_n\rangle = w_i |q w_1, \ldots, q w_{i-1}, w_i, \ldots, w_n\rangle. \tag{19}$$

In this case, the q-coherent state is not a coherent state because it is not a coherent state, which implies that it is not an eigenstate of annihilation operator. It comes from the fact that we adopted the ordinary complex variables as coherent variables and that a_i and $a_i^\dagger$ do not commute among themselves.

From this we find that it has the explicit form

$$\begin{aligned} |w\rangle &= U^\dagger(-w_1, \ldots, -w_n)|0\rangle \\ &= \left[\prod_{i=1}^{\infty} e_{q^2}^{-1/2}\left(|w_i|^2\right) \right] \sum_{n_1,\ldots,n_n}^{\infty} \frac{w_n^{n_n} \ldots w_1^{n_1}}{\sqrt{[n_n]! \cdots [n_1]!}} |n_1, \ldots, n_n\rangle. \end{aligned} \tag{20}$$

It can be easily checked that the above q-coherent states are normalized:

$$\langle w_1, \ldots, w_N | w_1, \ldots, w_N \rangle = 1. \tag{21}$$

The Perelomov-type coherent state is defined as the vacuum for the shifted q-annihilation operator

$$b_i|\bar{w}_1, \ldots, \bar{w}_N\rangle = 0, \tag{22}$$

where b_i is unitary transform of a_i;

$$b_i = U a_i U^\dagger, \tag{23}$$

where b_i is the unitary transform of a_i: From this we obtain

$$|\bar{w}_i\rangle = U(w_1, \ldots, w_n)|0\rangle = \sum_{n_1,\ldots,n_n}^{\infty} \left(\prod_{k=1}^{n} e_{q^2}^{-1/2}\left(|w_k|^2 q^{2\sum_{l=1}^{k} n_l}\right) \right)$$
$$\times \frac{\left(q^{n-1}w_1\right)^{n_1}\left(q^{n-2}w_2\right)^{n_2}\cdots(qw_{n-1})^{n_{n-1}} w_n^{n_n}}{\sqrt{[n_1]!\cdots[n_n]!}}|n_1, \ldots, n_n\rangle. \tag{24}$$

From Eqs. (20) and (24), we see the difference between the Glauber-type and Perelomov-type q-coherent states. The difference results from the fact that in the q-analogue of the unitary operator, the operator $U(\mathrm{w})$ and $U^\dagger(-\mathrm{w})$ are very different:

$$U(\mathrm{w}) \neq U^\dagger(-\mathrm{w}). \tag{25}$$

Therefore, these operators generate different sheaves of q-coherent states.

We recall that the eigenfunction problem for q-position operator satisfying

$$X_i\psi = q^{-\sum_{k=1}^{i-1} N_k}(a_i + a_i^\dagger)\psi = x_i\psi \tag{26}$$

generates the q-Hermite polynomials. Then, all q-position operators are commuting among themselves:

$$[X_i, X_j] = 0. \tag{27}$$

Let us expand ψ with respect to number eigenstate of algebra (1):

$$\psi = \sum_{n_1,\ldots,n_N}^{\infty} C_{n_1,\ldots,n_N}(x_1, \ldots, x_N)|n_1, \ldots, x_N\rangle \tag{28}$$

and write the expansion coefficients in the form

$$C_{n_1}, \ldots, n_N(x_1, \ldots, x_N) = C_0(x_1, \ldots, x_N)P_{n_1,\ldots,n_N}(x_1, \ldots, x_N). \tag{29}$$

Then, the function $P_{n_1,\ldots,n_N}$ satisfies the recurrence relation

$$\sqrt{[n_i + 1]}P_{n_1,\ldots,n_{i+1},\ldots,n_N} + \sqrt{[n_i]}P_{n_1,\ldots,n_{i-1},\ldots,n_N} = x_i P_{n_1,n_2,\ldots,n_N} \tag{30}$$

with initial condition $P_{0,0,\ldots,0} = 1$. Equation (30) shows that the n-variable function $P_{n_1,\ldots,n_N}$ is separable:

$$P_{n_1,\ldots,n_N} = \prod_{i=1}^{N} P_i(x_i). \tag{31}$$

Inserting Eq. (31) into Eq. (30) we have

$$\sqrt{[n_i+1]}P_{i+1} + \sqrt{[n_i]}P_{i-1} = x_i P_i. \tag{32}$$

The solution of Eq. (31) depends upon the value of q. When $q > 1$, it takes the form

$$P_i(x_i) = \frac{1}{\sqrt{(q^2;q^2)_{n_i}}} h_{n_i}\left(\frac{1}{2}(q^2-1)x_i \mid q^2\right), \tag{33}$$

where $h_n(x \mid q^2)$ is q-Hermite polynomial studied by Askey (1989). When $0 < q < 1$, it takes the form

$$P_i(x_i) = \frac{1}{\sqrt{(q^2;q^2)_{n_i}}} H_{n_i}\left(\frac{1}{2}(1-q^2)x_i \mid q^2\right), \tag{34}$$

where $H_n(x \mid q^2)$ is called the continuous q-Hermite polynomial (Askey and Ismail, 1983).

We now consider the eigenvalue problem for the shifted q-position operator. Since the number operator is not changed under the unitary transformation, the shifted q-position operator obeys

$$\tilde{X}_i\tilde{\psi} = q^{\sum_{k=1}^{i-1} N_k}(b_i + b_i^\dagger)\tilde{\psi} = x_i\tilde{\psi}. \tag{35}$$

Like Eq. (29), we can expand $\tilde{\psi}$ with repect to the number eigenstates as follows:

$$\tilde{\psi} = \sum_{n_1,\ldots,n_N}^{\infty} \tilde{C}_{n_1,\cdots,n_N}(x_1,\ldots,x_N)|n_{1,\ldots,}x_N\rangle \tag{36}$$

and write the expansion coefficients in the form

$$\tilde{C}_{n_1,\cdots,n_N}(x_1,\ldots,x_N) = \tilde{C}_0(x_1,\ldots,x_N)\tilde{P}_{n_1,\ldots,n_N}(x_1,\ldots,x_N). \tag{37}$$

Similarly we can factorize $\tilde{P}_{n_1,\ldots,n_N}(x_1,\ldots,x_N)$ as follows

$$\tilde{P}_{n_1,\ldots,n_N} = \prod_{i=1}^{N}\tilde{P}_i(x_i) \tag{38}$$

Inserting Eq. (38) into Eq. (35), we have the following three-term recurrence relation

$$\sqrt{[n_i+1]\big(1-|\dot{w}_i|^2(1-q^2)q^{2n_i}\big)}\,\tilde{P}_{i+1} - 2Re(w_i)q^{2ni}\tilde{P}_i$$

$$+\sqrt{[n_i]\left(1-|w_i|^2(1-q^2)q^{2n_i-2}\right)}\tilde{P}_{i-1} = x_i\tilde{P}_i. \tag{39}$$

Relation (29) again generates a certain system of orthogonal polynomials $\tilde{P}_i$, which is a kind of deformation of an ordinary Hermite polynomial depending on the parameter x_i. It is worth nothing that the family of polynomials $\tilde{P}(x_i; w_i)$ is isospectral, i.e. the spectrum x_i does not depend on the parameter w_i since

$$\tilde{\psi} = U(\mathrm{w})\psi. \tag{40}$$

In the classical limit $q \to 1$, we have

$$\tilde{P}_i(x_i; w_i) = H_{n_i}(x_i + 2Re(w_i)), \tag{41}$$

which implies that the unitary transform $U(\mathrm{w})$ is a shift of the force center of the oscillator.

To conclude, in this paper, I have studied two types of q-coherent states of $su_q(n)$-covariant oscillator algebra. One was the Glauber-type q-coherent state and another the Perelomov-type q-coherent state. In order to obtain the correct form of q-coherent states, I found out the unitary operator for $su_q(n)$-covariant oscillator algebra. As is different from the classical ($q \to 1$) case, the Glauber-type q-coherent state is not a coherent state in the ordinary sense because it is not an eigenstate of annihilation operator. It comes from the fact that the step operators of this algebra are noncommutative among themselves and that I used ordinary complex (not q-commuting) variables as coherent variables.

ACKNOWLEDGMENT

This work was supported by Korea Research Foundation Grant (KRF-2001-015-DS0006).

REFERENCES

Arik, M. and Coon, D. (1976). *Journal of Mathematical Physics* **17**, 524.

Askey, R. (1989). *q-Series and Partitions*, Springer, New York, pp. 151–158.

Askey, R. and Ismail, M. (1983). *Studies in Pure Mathematics*, Birkhaüser, Basel, pp. 55–58.

Biedenharn, L. (1989). *Journal of Physics A* **22**, L873.

Gasper, G. and Rhaman, M. (1990). *Basic Hypergeometric Series*, Cambridge University Press, Cambridge, UK.

Granovskii, Ya. I. and Zhedanov, A. (1986). *Izvestiya Yvysshikr Uchebnykh Zavednni Fizika* **5**, 60.

Jagannathan, R. et. al. (1992). *Journal of Physics A* **25**, 6429.

Macfarlane, A. (1989). *Journal of Physics A* **22**, 4581.

Nikiforov, A., Suslov, S., and Uvarov, V. (1985). *Classical Orthogonal Polynomials in a Discrete Variables* [in Russian], Nauka, Moscow.

Pusz, W. and Woronowicz, S. (1989). Reports on *Mathematical Physics* **27**, 231.